Satisfiability Modulo Theories

Clark Barrett, Stanford University

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Acknowledgments: Many thanks to Cesare Tinelli and Albert Oliveras for contributing some of the material used in these slides.

Disclamer: The literature on SMT and its applications is vast. The bibliographic references provided here are just a sample. Apologies to all authors whose work is not cited.

Introduction

A (Very) Brief History of Automated Reasoning

Philosophers have long dreamed of machines that can reason. The pursuit of this dream has occupied some of the best minds and led both to great acheivements and great disappointments.

~1700
Leibniz – mechanized human reasoning

1928
Hilbert

Entscheidungs-problem

1936
Church – lamda calculus

Turing – reduction halting problem

1954
Davis – decision procedure for Presburger arithmetic

Automated Reasoning: A Failure?

- At the turn of the century, automated reasoning was still considered by many to be impractical for most real-world applications
- Interesting problems appeared to be beyond the reach of automated methods because of decidability and complexity barriers
- The dream of *Hilbert*'s mechanized mathematics or *Leibniz*'s calculating machine was believed by many to be simply unattainable

The Satisfiability Revolution

Princeton, c. 2000

- *Chaff SAT solver*: orders of magnitude faster than previous SAT solvers
- *Important observation*: many real-world problems do not exhibit worst-case theoretical performance

Palo Alto, c. 2001

- Idea: combine fast new SAT solvers with decision procedures for decidable first-order theories
- *SVC*, *CVC* solvers (Stanford); *ICS*, *Yices* solvers (SRI)
- *Satisfiability Modulo Theories* (SMT) was born

SMT solvers

SMT solvers: *general-purpose* logic engines

- Given condition X, is it possible for Y to happen
- X and Y are expressed in a *rich logical language*
  - First-order logic
  - Domain-specific reasoning
    - arithmetic, arrays, bit-vectors, data types, etc.

SMT solvers are *changing the way people solve problems*

- Instead of building a *special-purpose* solver
- *Translate* into a logical formula and use an SMT solver
- Not only easier, *often better*

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A Security Example

Django

- Widely used open-source web development platform
- A security vulnerability in Django (CVE-2013-6044) was blamed on the following function[1]

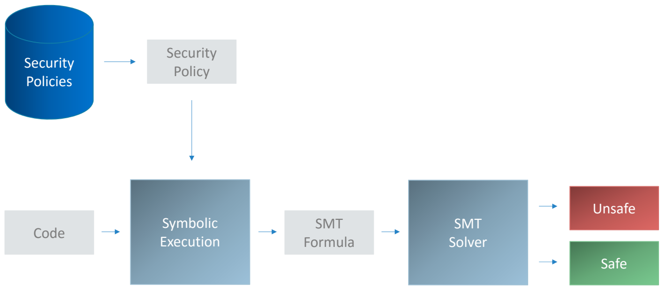
```python
def is_safe_url(url, host=None):
    """
    Return ``True`` if the url is a safe redirection (i.e. it doesn't
    point to a different host).

    Always returns ``False`` on an empty url.
    """
    if not url:
        return False
    netloc = urllib_parse.urlparse(url)[1]
    return not netloc or netloc == host
```

[1] https://github.com/django/django/blob/09a5f5aabe27f63ec8d8982efa6cef9bf7b86022/django/utils/http.py#L252

Using SMT To Find the Vulnerability

An approach for finding security vulnerabilities

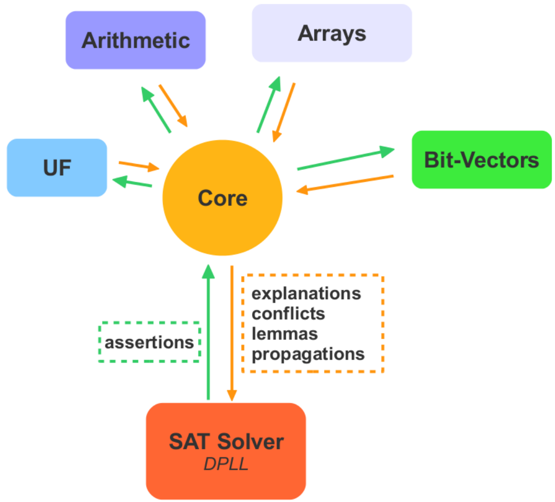
- *Symbolic execution*: generates a logical formula satisfiable iff code can violate security policy
- *SMT solver*: returns a solution or proves that none exists



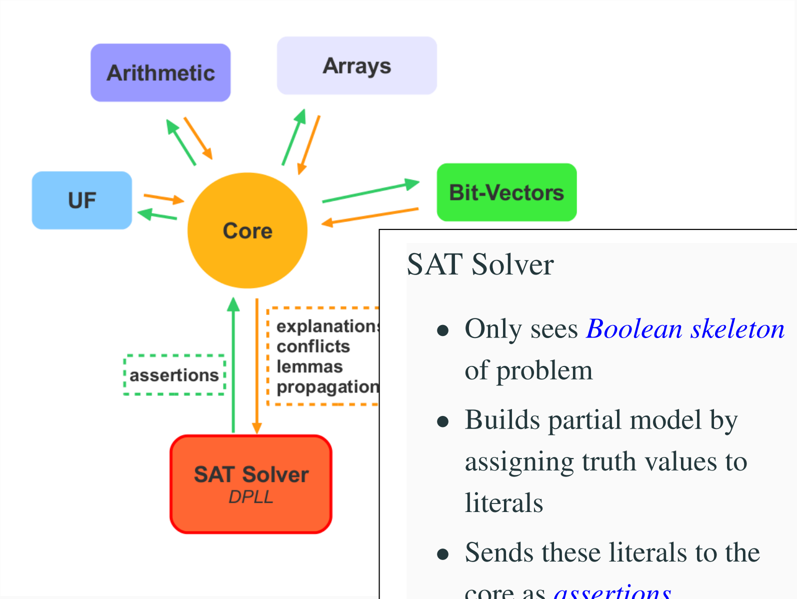
Using SMT To Find the Vulnerability

Demo: *Django XSS attack*

SMT Solvers


Arithmetic
Arrays
UF
Core
Bit-Vectors
assertions
explanations
conflicts
lemmas
propagations
SAT Solver
DPLL

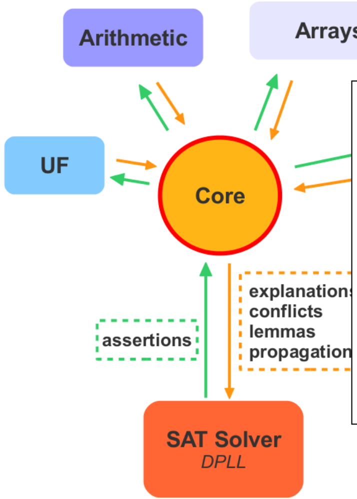
SMT Solvers



SAT Solver

- Only sees *Boolean skeleton* of problem
- Builds partial model by assigning truth values to literals
- Sends these literals to the core as *assertions*

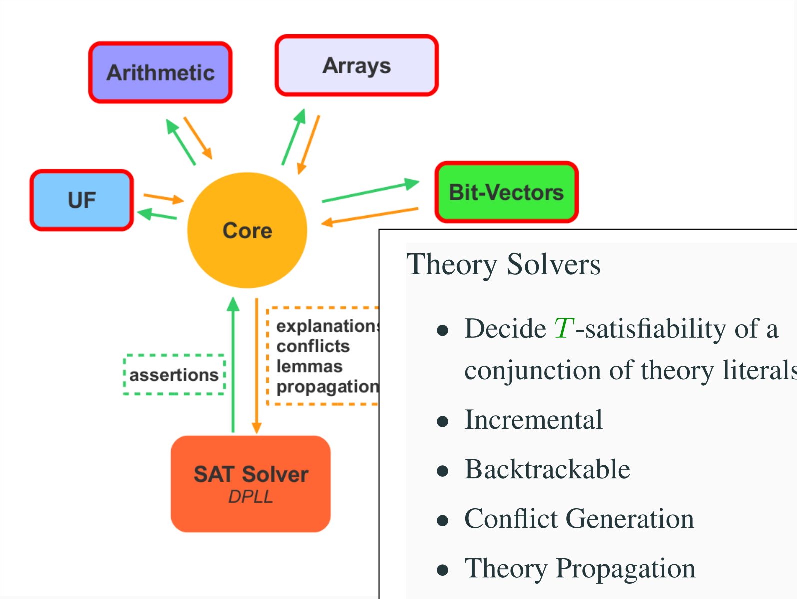
SMT Solvers



Core

- Sends each assertion to the appropriate theory
- Sends deduced literals to other theories/SAT solver
- Handles *theory combination*

SMT Solvers



Theory Solvers

- Decide T-satisfiability of a conjunction of theory literals
- Incremental
- Backtrackable
- Conflict Generation
- Theory Propagation

Theory Solvers

Theory Solvers

Given a theory T, a *Theory Solver* for T takes as input a set Φ of literals and determines whether Φ is T-satisfiable.

Φ is T-satisfiable iff there is some model M of T such that each formula in Φ holds in M.

Theories of Interest: UF

Equality ($=$) with Uninterpreted Functions [NO80, BD94, NO07]

Typically used to abstract unsupported constructs, e.g.:

- non-linear multiplication in arithmetic
- ALUs in circuits

Example: The formula

$$a * (|b| + c) = d \quad \wedge \quad b * (|a| + c) \neq d \quad \wedge \quad a = b$$

is unsatisfiable, but no arithmetic reasoning is needed

if we abstract it to

$$mul(a, add(abs(b), c)) = d \quad \wedge \quad mul(b, add(abs(a), c)) \neq d \quad \wedge \quad a = b$$

Theories of Interest: Arithmetic

Very useful, for obvious reasons

Restricted fragments (over the reals or the integers) support more efficient methods:

- Bounds: $x \bowtie k$ with $\bowtie \in \{<, >, \leq, \geq, =\}$ [BBC+05a]
- Difference logic: $x - y \bowtie k$, with $\bowtie \in \{<, >, \leq, \geq, =\}$ [NO05, WIGG05, CM06]
- UTVPI: $\pm x \pm y \bowtie k$, with $\bowtie \in \{<, >, \leq, \geq, =\}$ [LM05]
- Linear arithmetic, e.g: $2x - 3y + 4z \leq 5$ [DdM06]
- Non-linear arithmetic, e.g: $2xy + 4xz^2 - 5y \leq 10$ [BLNM+09, ZM10, JdM12]

Theories of Interest: Arrays

Used in software verification and hardware verification (for memories) [SBDL01, BNO+08a, dMB09]

Two interpreted function symbols read and write

Axiomatized by:

- $\forall a \, \forall i \, \forall v \; \mathrm{read}(\mathrm{write}(a, i, v), i) = v$
- $\forall a \, \forall i \, \forall j \, \forall v \; i \neq j \rightarrow \mathrm{read}(\mathrm{write}(a, i, v), j) = \mathrm{read}(a, j)$

Sometimes also with *extensionality*:

- $\forall a \, \forall b \; (\forall i \, \mathrm{read}(a, i) = \mathrm{read}(b, i) \; \rightarrow a = b)$

Is the following set of literals satisfiable in this theory?

$$\mathrm{write}(a, i, x) \neq b, \; \mathrm{read}(b, i) = y, \; \mathrm{read}(\mathrm{write}(b, i, x), j) = y, \; a = b, \; i = j$$

Theories of Interest: Bitvectors

Useful both in hardware and software verification [BCF+07, BB09, HBJ+14]

Universe consists of (fixed-sized) vectors of bits

Different types of operations:

- *String-like*: concat, extract, …
- *Logical*: bit-wise not, or, and, …
- *Arithmetic*: add, subtract, multiply, …
- *Comparison*: $<, >, \ldots$

Is this formula satisfiable over bitvectors of size 3?

$$a[1:0] \neq b[1:0] \ \wedge \ (a \mid b) = c \ \wedge \ c[0] = 0 \ \wedge \ a[1] + b[1] = 0$$

Implementing a Theory Solver: Difference Logic

We consider a simple example: difference logic.

In *difference logic*, we are interested in the satisfiability of a conjunction of arithmetic atoms.

Each atom is of the form $x - y \bowtie c$, where x and y are variables, c is a numeric constant, and $\bowtie \in \{=, <, \leq, >, \geq\}$.

The variables can range over either the *integers* (QF_IDL) or the *reals* (QF_RDL).

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- $x - y < c \quad \Longrightarrow \quad x - y \leq c - 1$ (integers)

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- $x - y < c \quad \implies \quad x - y \leq c - 1$ (integers)
- $x - y < c \quad \implies \quad x - y \leq c - \delta$ (reals)

Difference Logic

Now we have a conjunction of literals, all of the form $x - y \leq c$.

From these literals, we form a weighted directed graph with a vertex for each variable.

For each literal $x - y \leq c$, there is an edge $x \xrightarrow{c} y$.

The set of literals is satisfiable iff there is no cycle for which the sum of the weights on the edges is negative.

There are a number of efficient algorithms for detecting negative cycles in graphs.

Difference Logic Example

$$x - y = 5 \ \wedge \ z - y \geq 2 \ \wedge \ z - x > 2 \ \wedge \ w - x = 2 \ \wedge \ z - w < 0$$

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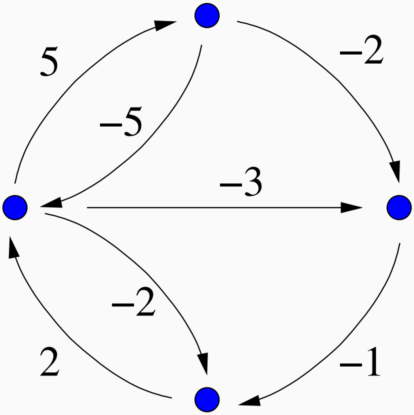
$$\begin{array}{lcl}
x - y = 5 & & x - y \leq 5 \wedge y - x \leq -5 \\
z - y \geq 2 & & y - z \leq -2 \\
z - x > 2 & \Rightarrow & x - z \leq -3 \\
w - x = 2 & & w - x \leq 2 \wedge x - w \leq -2 \\
z - w < 0 & & z - w \leq -1
\end{array}$$

Difference Logic Example



DPLL(T): Combining T-Solvers with SAT

Satisfiability Modulo a Theory T

Def. A formula is *(un)satisfiable in* a theory T, or T-*(un)satisfiable*, if there is a (no) model of T that satisfies it

Note: The T-satisfiability of quantifier-free formulas is decidable iff the T-satisfiability of conjunctions/sets of literals is decidable

(Convert the formula in DNF and check if any of its disjuncts is T-sat)

Problem: In practice, dealing with Boolean combinations of literals is as hard as in propositional logic

Solution: Exploit propositional satisfiability technology

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Lifting SAT Technology to SMT

Two main approaches:

1. "Eager" [PRSS99, SSB02, SLB03, BGV01, BV02]
   - translate into an equisatisfiable propositional formula
   - feed it to any SAT solver

Notable systems: *UCLID*

2. "Lazy" [ACG00, dMR02, BDS02, ABC+02]
   - abstract the input formula to a propositional one
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   - use a theory decision procedure to refine the formula and guide the SAT solver

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(Very) Lazy Approach for SMT – Example

$$g(a) = c \quad \wedge \quad f(g(a)) \neq f(c) \vee g(a) = d \quad \wedge \quad c \neq d$$

Theory T: Equality with Uninterpreted Functions

Simplest setting:

- Off-line SAT solver
- Non-incremental *theory solver* for conjunctions of equalities and disequalities
- Theory atoms (e.g., $g(a) = c$) abstracted to propositional atoms (e.g., 1)

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- SAT solver returns model $\{1,\ \overline{2},\ \overline{4}\}$.
  Theory solver finds (concretization of) $\{1,\ \overline{2},\ \overline{4}\}$ unsat.
- Send $\{1,\ \overline{2} \vee 3,\ \overline{4},\ \overline{1} \vee 2 \vee 4\}$ to SAT solver.
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  Done: the original formula is unsatisfiable in UF.

Lazy Approach – Enhancements

Several enhancements are possible to increase efficiency:

- Check T-satisfiability only of full propositional model
- Check T-satisfiability of partial assignment M as it grows
- If M is T-unsatisfiable, identify a T-unsatisfiable subset M_0 of M and add $\neg M_0$ as a clause
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Lazy Approach – Main Benefits

- Every tool does what it is good at:
  - SAT solver takes care of Boolean information
  - Theory solver takes care of theory information
- The theory solver works only with conjunctions of literals
- Modular approach:
  - SAT and theory solvers communicate via a simple API [GHN+04]
  - SMT for a new theory only requires new theory solver
  - An off-the-shelf SAT solver can be embedded in a lazy SMT system with few new lines of code (tens)

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An Abstract Framework for Lazy SMT

Several variants and enhancements of lazy SMT solvers exist

They can be modeled abstractly and declaratively as *transition systems*

A transition system is a binary relation over states, induced by a set of conditional transition rules

The framework can be first developed for SAT and then extended to lazy SMT [NOT06, KG07]

Advantages of Abstract Framework

An abstract framework helps one:

- skip over implementation details and unimportant control aspects
- reason formally about solvers for SAT and SMT
- model advanced features such as non-chronological bactracking, lemma learning, theory propagation, ...
- describe different strategies and prove their correctness
- compare different systems at a higher level
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The one described next is a re-elaboration of those in

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The Original DPLL Procedure

- Modern SAT solvers are based on the DPLL procedure [DP60, DLL62]
- DPLL tries to build incrementally a satisfying truth assignment M for a CNF formula F
- M is grown by
    - deducing the truth value of a literal from M and F, or
    - guessing a truth value
- If a wrong guess for a literal leads to an inconsistency, the procedure backtracks and tries the opposite value

An Abstract Framework for DPLL

States:

$$\mathsf{fail} \quad \text{or} \quad \langle M, F \rangle$$

where

- M is a sequence of literals and *decision points* $\bullet$ denoting a partial truth *assignment*
- F is a set of clauses denoting a CNF *formula*

Def. If $M = M_0 \bullet M_1 \bullet \cdots \bullet M_n$ where each M_i contains no decision points

- M_i is *decision level* i of M
- $M^{[i]} \stackrel{\text{def}}{=} M_0 \bullet \cdots \bullet M_i$

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Initial state:

- $\langle (), F_0\rangle$, where F_0 is to be checked for satisfiability

Expected final states:

- fail if F_0 is unsatisfiable
- $\langle M, G\rangle$ otherwise, where
  - G is equivalent to F_0 and
  - M satisfies G

Transition Rules: Notation

States treated like records:

- M denotes the truth assignment component of current state
- F denotes the formula component of current state

Transition rules in *guarded assignment form* [KG07]

$$\frac{p_1 \quad \cdots \quad p_n}{[\mathsf{M} := e_1] \quad [\mathsf{F} := e_2]}$$

updating M, F or both when premises $p_1, \ldots, p_n$ all hold

Transition Rules for the Original DPLL

Extending the assignment

Propagate $$\frac{l_1 \vee \cdots \vee l_n \vee l \in \mathsf{F} \quad \bar{l}_1, \ldots, \bar{l}_n \in \mathsf{M} \quad l, \bar{l} \notin \mathsf{M}}{\mathsf{M} := \mathsf{M}\ l}$$

Note: When convenient, treat M as a set

Note: Clauses are treated modulo ACI of $\vee$

Decide $$\frac{l \in \mathrm{Lit}(\mathsf{F}) \quad l, \bar{l} \notin \mathsf{M}}{\mathsf{M} := \mathsf{M} \bullet l}$$

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Backtrack

$$\frac{l_1 \vee \cdots \vee l_n \in \mathsf{F} \quad \bar{l}_1, \ldots, \bar{l}_n \in \mathsf{M} \quad \mathsf{M} = M \bullet l\, N \quad \bullet \notin N}{\mathsf{M} := M\, \bar{l}}$$

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From DPLL to CDCL Solvers (1)

To model conflict-driven backjumping and learning, add to states a third component C whose value is either no or a *conflict clause*

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From DPLL to CDCL Solvers (2)

Replace **Backtrack** with

Conflict $$\frac{\mathsf{C} = \mathsf{no} \quad l_1 \vee \cdots \vee l_n \in \mathsf{F} \quad \bar{l}_1, \ldots, \bar{l}_n \in \mathsf{M}}{\mathsf{C} := l_1 \vee \cdots \vee l_n}$$

Explain $$\frac{\mathsf{C} = l \vee D \quad l_1 \vee \cdots \vee l_n \vee \bar{l} \in \mathsf{F} \quad \bar{l}_1, \ldots, \bar{l}_n \prec_{\mathsf{M}} \bar{l}}{\mathsf{C} := l_1 \vee \cdots \vee l_n \vee D}$$

Backjump $$\frac{\mathsf{C} = l_1 \vee \cdots \vee l_n \vee l \quad \mathsf{lev}\ \bar{l}_1, \ldots, \mathsf{lev}\ \bar{l}_n \leq \ i < \mathsf{lev}\ \bar{l}}{\mathsf{C} := \mathsf{no} \quad \mathsf{M} := \mathsf{M}^{[i]}\ l}$$

Maintain invariant: $\mathsf{F} \models_{\mathrm{p}} \mathsf{C}$ and $\mathsf{M} \models_{\mathrm{p}} \neg\mathsf{C}$ when $\mathsf{C} \neq \mathsf{no}$

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Note: $l \prec_{\mathsf{M}} l'$ if l occurs before l' in M
$\mathsf{lev}\ l = i$ iff l occurs in decision level i of M

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From DPLL to CDCL Solvers (3)

Modify **Fail** to

$$\textbf{Fail}\ \frac{C \neq \mathsf{no} \quad \bullet \notin M}{\mathsf{fail}}$$

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Execution Example

$$F := \{1,\ \overline{1} \vee 2,\ \overline{3} \vee 4,\ \overline{5} \vee \overline{6},\ \overline{1} \vee \overline{5} \vee 7,\ \overline{2} \vee \overline{5} \vee 6 \vee \overline{7}\}$$

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$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}\ 7$	F	$\overline{1} \vee \overline{2} \vee \overline{5}$	by **Explain** with $\overline{5} \vee \overline{6}$
$1\ 2\ \overline{5}$	F	no	by **Backjump**
$1\ 2\ \overline{5} \bullet 3$	F	no	by **Decide**
$\ldots$			

Execution Example

$$F := \{1,\ \ \overline{1} \vee 2,\ \ \overline{3} \vee 4,\ \ \overline{5} \vee \overline{6},\ \ \overline{1} \vee \overline{5} \vee 7,\ \ \overline{2} \vee \overline{5} \vee 6 \vee \overline{7}\}$$

M	F	C	rule
	F	no	
1	F	no	by **Propagate**
$1\ 2$	F	no	by **Propagate**
$1\ 2 \bullet 3$	F	no	by **Decide**
$1\ 2 \bullet 3\ 4$	F	no	by **Propagate**
$1\ 2 \bullet 3\ 4 \bullet 5$	F	no	by **Decide**
$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}$	F	no	by **Propagate**
$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}\ 7$	F	no	by **Propagate**
$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}\ 7$	F	$\overline{2} \vee \overline{5} \vee 6 \vee \overline{7}$	by **Conflict**
$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}\ 7$	F	$\overline{1} \vee 2 \vee \overline{5} \vee 6$	by **Explain** with $\overline{1} \vee \overline{5} \vee 7$
$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}\ 7$	F	$\overline{1} \vee \overline{2} \vee \overline{5}$	by **Explain** with $\overline{5} \vee \overline{6}$
$1\ 2\ \overline{5}$	F	no	by **Backjump**
$1\ 2\ \overline{5} \bullet 3$	F	no	by **Decide**
$\dots$			

Execution Example

$$F := \{1,\ \ \overline{1} \vee 2,\ \ \overline{3} \vee 4,\ \ \overline{5} \vee \overline{6},\ \ \overline{1} \vee \overline{5} \vee 7,\ \ \overline{2} \vee \overline{5} \vee 6 \vee \overline{7}\}$$

M	F	C	rule
	F	no	
1	F	no	by **Propagate**
$1\ 2$	F	no	by **Propagate**
$1\ 2 \bullet 3$	F	no	by **Decide**
$1\ 2 \bullet 3\ 4$	F	no	by **Propagate**
$1\ 2 \bullet 3\ 4 \bullet 5$	F	no	by **Decide**
$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}$	F	no	by **Propagate**
$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}\ 7$	F	no	by **Propagate**
$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}\ 7$	F	$\overline{2} \vee \overline{5} \vee 6 \vee \overline{7}$	by **Conflict**
$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}\ 7$	F	$\overline{1} \vee \overline{2} \vee \overline{5} \vee 6$	by **Explain** with $\overline{1} \vee \overline{5} \vee 7$
$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}\ 7$	F	$\overline{1} \vee \overline{2} \vee \overline{5}$	by **Explain** with $\overline{5} \vee \overline{6}$
$1\ 2\ \overline{5}$	F	no	by **Backjump**
$1\ 2\ \overline{5} \bullet 3$	F	no	by **Decide**
$\ldots$			

Execution Example

$$F := \{1,\ \overline{1} \vee 2,\ \overline{3} \vee 4,\ \overline{5} \vee \overline{6},\ \overline{1} \vee \overline{5} \vee 7,\ \overline{2} \vee \overline{5} \vee 6 \vee \overline{7}\}$$

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1	F	no	by **Propagate**
$1\ 2$	F	no	by **Propagate**
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$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}\ 7$	F	no	by **Propagate**
$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}\ 7$	F	$\overline{2} \vee \overline{5} \vee 6 \vee \overline{7}$	by **Conflict**
$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}\ 7$	F	$\overline{1} \vee \overline{2} \vee \overline{5} \vee 6$	by **Explain** with $\overline{1} \vee \overline{5} \vee 7$
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$1\ 2\ \overline{5}$	F	no	by **Backjump**
$1\ 2\ \overline{5} \bullet 3$	F	no	by **Decide**
$\dots$			

Execution Example

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$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}$	F	no	by **Propagate**
$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}\ 7$	F	no	by **Propagate**
$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}\ 7$	F	$\overline{2} \vee \overline{5} \vee 6 \vee \overline{7}$	by **Conflict**
$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}\ 7$	F	$\overline{1} \vee \overline{2} \vee \overline{5} \vee 6$	by **Explain** with $\overline{1} \vee \overline{5} \vee 7$
$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}\ 7$	F	$\overline{1} \vee \overline{2} \vee \overline{5}$	by **Explain** with $\overline{5} \vee \overline{6}$
$1\ 2\ \overline{5}$	F	no	by **Backjump**
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$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}\ 7$	F	$\overline{1} \vee \overline{2} \vee \overline{5} \vee 6$	by **Explain** with $\overline{1} \vee \overline{5} \vee 7$
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$1\ 2\ \overline{5}$	F	no	by **Backjump**
$1\ 2\ \overline{5} \bullet 3$	F	no	by **Decide**
$\dots$			

From DPLL to CDCL Solvers (4)

Also add

$$\textbf{Learn}\quad \frac{\mathsf{F} \models_{\mathrm{p}} C \qquad C \notin \mathsf{F}}{\mathsf{F} := \mathsf{F} \cup \{C\}}$$

$$\textbf{Forget}\quad \frac{\mathsf{C} = \mathsf{no} \qquad \mathsf{F} = G \cup \{C\} \qquad G \models_{\mathrm{p}} C}{\mathsf{F} := G}$$

$$\textbf{Restart}\quad \frac{}{\mathsf{M} := \mathsf{M}^{[0]} \qquad \mathsf{C} := \mathsf{no}}$$

Note: Learn can be applied to any clause stored in C when $\mathsf{C} \neq \mathsf{no}$

Modeling Modern SAT Solvers

At the core, current CDCL SAT solvers are implementations of the transition system with rules

Propagate, **Decide**,

Conflict, **Explain**, **Backjump**,

Learn, **Forget**, **Restart**

Basic DPLL $\stackrel{\text{def}}{=}$

{ **Propagate**, **Decide**, **Conflict**, **Explain**, **Backjump** }

DPLL $\stackrel{\text{def}}{=}$ Basic DPLL + { **Learn**, **Forget**, **Restart** }

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The Basic DPLL System – Correctness

Some terminology:

Irreducible state: state for which no Basic DPLL rules apply

Execution: sequence of transitions allowed by the rules and starting with $\mathsf{M} = \emptyset$ and $\mathsf{C} = \mathsf{no}$

Exhausted execution: execution ending in an irreducible state

Proposition (Soundness) For every exhausted execution starting with $\mathsf{F} = F_0$ and ending with fail, the clause set F_0 is unsatisfiable.

Proposition (Completeness) For every exhausted execution starting with $\mathsf{F} = F_0$ and ending with $\mathsf{C} = \mathsf{no}$, the clause set F_0 is satisfied by M.

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Proposition (Strong Termination) Every execution in Basic DPLL is finite.

Note: This is not so immediate, because of **Backjump**.

Proposition (Soundness) For every exhausted execution starting with $\mathsf{F} = F_0$ and ending with fail, the clause set F_0 is unsatisfiable.

Proposition (Completeness) For every exhausted execution starting

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Exhausted execution: execution ending in an irreducible state

Proposition (Strong Termination) Every execution in Basic DPLL is finite.

Lemma Every exhausted execution ends with either $\mathsf{C} = \mathsf{no}$ or fail.

Proposition (Soundness) For every exhausted execution starting with $\mathsf{F} = F_0$ and ending with fail, the clause set F_0 is unsatisfiable.

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The DPLL System – Strategies

- Applying
  - one Basic DPLL rule between each two **Learn** applications and
  - **Restart** less and less often

  ensures termination

- A common basic strategy applies the rules with the following priorities:
  1. If $n > 0$ conflicts have been found so far, increase n and apply **Restart**
  2. If a clause is falsified by M, apply **Conflict**
  3. Keep applying **Explain** until **Backjump** is applicable
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  6. Apply **Propagate** to completion
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  3. Keep applying **Explain** until **Backjump** is applicable
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  5. Apply **Backjump**
  6. Apply **Propagate** to completion
  7. Apply **Decide**

From SAT to SMT

Same states and transitions but

- F contains quantifier-free clauses in some theory T
- M is a sequence of theory literals and decision points
- the DPLL system is augmented with rules

  T**-Conflict**, T**-Propagate**, T**-Explain**

- maintains invariant: $\mathsf{F} \models_T \mathsf{C}$ and $\mathsf{M} \models_{\mathrm{p}} \neg\mathsf{C}$ when $\mathsf{C} \neq \mathsf{no}$

Def. $F \models_T G$ iff every model of T that satisfies F satisfies G as well

SMT-level Rules

Fix a theory T

T-Conflict $$\frac{\mathsf{C} = \mathsf{no} \quad l_1, \ldots, l_n \in \mathsf{M} \quad l_1, \ldots, l_n \models_T \bot}{\mathsf{C} := \bar{l}_1 \vee \cdots \vee \bar{l}_n}$$

T-Propagate $$\frac{l \in \mathrm{Lit}(\mathsf{F}) \quad \mathsf{M} \models_T l \quad l, \bar{l} \notin \mathsf{M}}{\mathsf{M} := \mathsf{M}\ l}$$

T-Explain $$\frac{\mathsf{C} = l \vee D \quad \bar{l}_1, \ldots, \bar{l}_n \models_T \bar{l} \quad \bar{l}_1, \ldots, \bar{l}_n \prec_{\mathsf{M}} \bar{l}}{\mathsf{C} := l_1 \vee \cdots \vee l_n \vee D}$$

Note: $\bot$ = empty clause

Note: $\models_T$ decided by theory solver

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Note: $\bot$ = empty clause

Note: $\models_T$ decided by theory solver

Modeling the Very Lazy Theory Approach

T**-Conflict** is enough to model the naive integration of SAT solvers and theory solvers seen in the earlier UF example

$$\underbrace{g(a) = c}_{1} \quad \wedge \quad \underbrace{f(g(a)) \neq f(c)}_{\overline{2}} \vee \underbrace{g(a) = d}_{3} \quad \wedge \quad \underbrace{c \neq d}_{\overline{4}}$$

M	F	C	rule
	$1,\ \overline{2} \vee 3,\ \overline{4}$	no	
$1\ \overline{4}$	$1,\ \overline{2} \vee 3,\ \overline{4}$	no	by **Propagate**$^+$
$1\ \overline{4} \bullet \overline{2}$	$1,\ \overline{2} \vee 3,\ \overline{4}$	no	by **Decide**
$1\ \overline{4} \bullet \overline{2}$	$1,\ \overline{2} \vee 3,\ \overline{4}$	$\overline{1} \vee 2 \vee 4$	by T**-Conflict**
$1\ \overline{4} \bullet \overline{2}$	$1,\ \overline{2} \vee 3,\ \overline{4},\ \overline{1} \vee 2 \vee 4$	$\overline{1} \vee 2 \vee 4$	by **Learn**
$1\ \overline{4}$	$1,\ \overline{2} \vee 3,\ \overline{4},\ \overline{1} \vee 2 \vee 4$	no	by **Restart**
$1\ \overline{4}\ 2\ 3$	$1,\ \overline{2} \vee 3,\ \overline{4},\ \overline{1} \vee 2 \vee 4$	no	by **Propagate**$^+$
$1\ \overline{4}\ 2\ 3$	$1,\ \overline{2} \vee 3,\ \overline{4},\ \overline{1} \vee 2 \vee 4,\ \overline{1} \vee \overline{3} \vee 4$	$\overline{1} \vee \overline{3} \vee 4$	by T**-Conflict**, **Learn**
fail			by **Fail**

Modeling the Very Lazy Theory Approach

$$\underbrace{g(a) = c}_{1} \quad \wedge \quad \underbrace{f(g(a)) \neq f(c)}_{\overline{2}} \vee \underbrace{g(a) = d}_{3} \quad \wedge \quad \underbrace{c \neq d}_{\overline{4}}$$

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$1\ \overline{4} \bullet \overline{2}$	$1,\ \overline{2} \vee 3,\ \overline{4}$	no	by **Decide**
$1\ \overline{4} \bullet \overline{2}$	$1,\ \overline{2} \vee 3,\ \overline{4}$	$\overline{1} \vee 2 \vee 4$	by **T-Conflict**
$1\ \overline{4} \bullet \overline{2}$	$1,\ \overline{2} \vee 3,\ \overline{4},\ \overline{1} \vee 2 \vee 4$	$\overline{1} \vee 2 \vee 4$	by **Learn**
$1\ \overline{4}$	$1,\ \overline{2} \vee 3,\ \overline{4},\ \overline{1} \vee 2 \vee 4$	no	by **Restart**
$1\ \overline{4}\ 2\ 3$	$1,\ \overline{2} \vee 3,\ \overline{4},\ \overline{1} \vee 2 \vee 4$	no	by **Propagate**$^+$
$1\ \overline{4}\ 2\ 3$	$1,\ \overline{2} \vee 3,\ \overline{4},\ \overline{1} \vee 2 \vee 4,\ \overline{1} \vee \overline{3} \vee 4$	$\overline{1} \vee \overline{3} \vee 4$	by **T-Conflict**, **Learn**
fail			by **Fail**

Modeling the Very Lazy Theory Approach

$$\underbrace{g(a) = c}_{1} \quad \wedge \quad \underbrace{f(g(a)) \neq f(c)}_{\overline{2}} \vee \underbrace{g(a) = d}_{3} \quad \wedge \quad \underbrace{c \neq d}_{\overline{4}}$$

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$1\ \overline{4} \bullet \overline{2}$	$1,\ \overline{2} \vee 3,\ \overline{4}$	no	by **Decide**
$1\ \overline{4} \bullet \overline{2}$	$1,\ \overline{2} \vee 3,\ \overline{4}$	$\overline{1} \vee 2 \vee 4$	by T**-Conflict**
$1\ \overline{4} \bullet \overline{2}$	$1,\ \overline{2} \vee 3,\ \overline{4},\ \overline{1} \vee 2 \vee 4$	$\overline{1} \vee 2 \vee 4$	by **Learn**
$1\ \overline{4}$	$1,\ \overline{2} \vee 3,\ \overline{4},\ \overline{1} \vee 2 \vee 4$	no	by **Restart**
$1\ \overline{4}\ 2\ 3$	$1,\ \overline{2} \vee 3,\ \overline{4},\ \overline{1} \vee 2 \vee 4$	no	by **Propagate**$^+$
$1\ \overline{4}\ 2\ 3$	$1,\ \overline{2} \vee 3,\ \overline{4},\ \overline{1} \vee 2 \vee 4,\ \overline{1} \vee \overline{3} \vee 4$	$\overline{1} \vee \overline{3} \vee 4$	by T**-Conflict**, **Learn**
fail			by **Fail**

Modeling the Very Lazy Theory Approach

$$\underbrace{g(a) = c}_{1} \quad \wedge \quad \underbrace{f(g(a)) \neq f(c)}_{\overline{2}} \vee \underbrace{g(a) = d}_{3} \quad \wedge \quad \underbrace{c \neq d}_{\overline{4}}$$

M	F	C	rule
	$1,\ \overline{2} \vee 3,\ \overline{4}$	no	
$1\ \overline{4}$	$1,\ \overline{2} \vee 3,\ \overline{4}$	no	by **Propagate**$^+$
$1\ \overline{4} \bullet \overline{2}$	$1,\ \overline{2} \vee 3,\ \overline{4}$	no	by **Decide**
$1\ \overline{4} \bullet \overline{2}$	$1,\ \overline{2} \vee 3,\ \overline{4}$	$\overline{1} \vee 2 \vee 4$	by ***T*-Conflict**
$1\ \overline{4} \bullet \overline{2}$	$1,\ \overline{2} \vee 3,\ \overline{4},\ \overline{1} \vee 2 \vee 4$	$\overline{1} \vee 2 \vee 4$	by **Learn**
$1\ \overline{4}$	$1,\ \overline{2} \vee 3,\ \overline{4},\ \overline{1} \vee 2 \vee 4$	no	by **Restart**
$1\ \overline{4}\ 2\ 3$	$1,\ \overline{2} \vee 3,\ \overline{4},\ \overline{1} \vee 2 \vee 4$	no	by **Propagate**$^+$
$1\ \overline{4}\ 2\ 3$	$1,\ \overline{2} \vee 3,\ \overline{4},\ \overline{1} \vee 2 \vee 4,\ \overline{1} \vee \overline{3} \vee 4$	$\overline{1} \vee \overline{3} \vee 4$	by ***T*-Conflict**, **Learn**
fail			by **Fail**

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$$\underbrace{g(a) = c}_{1} \quad \wedge \quad \underbrace{f(g(a)) \neq f(c)}_{\overline{2}} \vee \underbrace{g(a) = d}_{3} \quad \wedge \quad \underbrace{c \neq d}_{\overline{4}}$$

M	F	C	rule
	$1,\ \overline{2} \vee 3,\ \overline{4}$	no	
$1\ \overline{4}$	$1,\ \overline{2} \vee 3,\ \overline{4}$	no	by **Propagate**$^+$
$1\ \overline{4} \bullet \overline{2}$	$1,\ \overline{2} \vee 3,\ \overline{4}$	no	by **Decide**
$1\ \overline{4} \bullet \overline{2}$	$1,\ \overline{2} \vee 3,\ \overline{4}$	$\overline{1} \vee 2 \vee 4$	by ***T*-Conflict**
$1\ \overline{4} \bullet \overline{2}$	$1,\ \overline{2} \vee 3,\ \overline{4},\ \overline{1} \vee 2 \vee 4$	$\overline{1} \vee 2 \vee 4$	by **Learn**
$1\ \overline{4}$	$1,\ \overline{2} \vee 3,\ \overline{4},\ \overline{1} \vee 2 \vee 4$	no	by **Restart**
$1\ \overline{4}\ 2\ 3$	$1,\ \overline{2} \vee 3,\ \overline{4},\ \overline{1} \vee 2 \vee 4$	no	by **Propagate**$^+$
$1\ \overline{4}\ 2\ 3$	$1,\ \overline{2} \vee 3,\ \overline{4},\ \overline{1} \vee 2 \vee 4,\ \overline{1} \vee \overline{3} \vee 4$	$\overline{1} \vee \overline{3} \vee 4$	by ***T*-Conflict**, **Learn**
fail			by **Fail**

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M	F	C	rule
	$1,\ \overline{2} \vee 3,\ \overline{4}$	no	
$1\ \overline{4}$	$1,\ \overline{2} \vee 3,\ \overline{4}$	no	by **Propagate**$^{+}$
$1\ \overline{4} \bullet \overline{2}$	$1,\ \overline{2} \vee 3,\ \overline{4}$	no	by **Decide**
$1\ \overline{4} \bullet \overline{2}$	$1,\ \overline{2} \vee 3,\ \overline{4}$	$\overline{1} \vee 2 \vee 4$	by **T-Conflict**
$1\ \overline{4} \bullet \overline{2}$	$1,\ \overline{2} \vee 3,\ \overline{4},\ \overline{1} \vee 2 \vee 4$	$\overline{1} \vee 2 \vee 4$	by **Learn**
$1\ \overline{4}$	$1,\ \overline{2} \vee 3,\ \overline{4},\ \overline{1} \vee 2 \vee 4$	no	by **Restart**
$1\ \overline{4}\ 2\ 3$	$1,\ \overline{2} \vee 3,\ \overline{4},\ \overline{1} \vee 2 \vee 4$	no	by **Propagate**$^{+}$
$1\ \overline{4}\ 2\ 3$	$1,\ \overline{2} \vee 3,\ \overline{4},\ \overline{1} \vee 2 \vee 4,\ \overline{1} \vee \overline{3} \vee 4$	$\overline{1} \vee \overline{3} \vee 4$	by **T-Conflict, Learn**
fail			by **Fail**

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	$1,\ \overline{2} \vee 3,\ \overline{4}$	no	
$1\ \overline{4}$	$1,\ \overline{2} \vee 3,\ \overline{4}$	no	by **Propagate**$^{+}$
$1\ \overline{4} \bullet \overline{2}$	$1,\ \overline{2} \vee 3,\ \overline{4}$	no	by **Decide**
$1\ \overline{4} \bullet \overline{2}$	$1,\ \overline{2} \vee 3,\ \overline{4}$	$\overline{1} \vee 2 \vee 4$	by **T-Conflict**
$1\ \overline{4} \bullet \overline{2}$	$1,\ \overline{2} \vee 3,\ \overline{4},\ \overline{1} \vee 2 \vee 4$	$\overline{1} \vee 2 \vee 4$	by **Learn**
$1\ \overline{4}$	$1,\ \overline{2} \vee 3,\ \overline{4},\ \overline{1} \vee 2 \vee 4$	no	by **Restart**
$1\ \overline{4}\ 2\ 3$	$1,\ \overline{2} \vee 3,\ \overline{4},\ \overline{1} \vee 2 \vee 4$	no	by **Propagate**$^{+}$
$1\ \overline{4}\ 2\ 3$	$1,\ \overline{2} \vee 3,\ \overline{4},\ \overline{1} \vee 2 \vee 4,\ \overline{1} \vee \overline{3} \vee 4$	$\overline{1} \vee \overline{3} \vee 4$	by **T-Conflict, Learn**
fail			by **Fail**

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M	F	C	rule
	$1,\ \overline{2} \vee 3,\ \overline{4}$	no	
$1\ \overline{4}$	$1,\ \overline{2} \vee 3,\ \overline{4}$	no	by **Propagate**$^+$
$1\ \overline{4} \bullet \overline{2}$	$1,\ \overline{2} \vee 3,\ \overline{4}$	no	by **Decide**
$1\ \overline{4} \bullet \overline{2}$	$1,\ \overline{2} \vee 3,\ \overline{4}$	$\overline{1} \vee 2 \vee 4$	by ***T*-Conflict**
$1\ \overline{4} \bullet \overline{2}$	$1,\ \overline{2} \vee 3,\ \overline{4},\ \overline{1} \vee 2 \vee 4$	$\overline{1} \vee 2 \vee 4$	by **Learn**
$1\ \overline{4}$	$1,\ \overline{2} \vee 3,\ \overline{4},\ \overline{1} \vee 2 \vee 4$	no	by **Restart**
$1\ \overline{4}\ 2\ 3$	$1,\ \overline{2} \vee 3,\ \overline{4},\ \overline{1} \vee 2 \vee 4$	no	by **Propagate**$^+$
$1\ \overline{4}\ 2\ 3$	$1,\ \overline{2} \vee 3,\ \overline{4},\ \overline{1} \vee 2 \vee 4,\ \overline{1} \vee \overline{3} \vee 4$	$\overline{1} \vee \overline{3} \vee 4$	by ***T*-Conflict**, **Learn**
fail			by **Fail**

Modeling the Very Lazy Theory Approach

$$\underbrace{g(a)=c}_{1} \quad \wedge \quad \underbrace{f(g(a)) \neq f(c)}_{\overline{2}} \vee \underbrace{g(a)=d}_{3} \quad \wedge \quad \underbrace{c \neq d}_{\overline{4}}$$

M	F	C	rule
	$1,\ \overline{2} \vee 3,\ \overline{4}$	no	
$1\ \overline{4}$	$1,\ \overline{2} \vee 3,\ \overline{4}$	no	by **Propagate**$^{+}$
$1\ \overline{4} \bullet \overline{2}$	$1,\ \overline{2} \vee 3,\ \overline{4}$	no	by **Decide**
$1\ \overline{4} \bullet \overline{2}$	$1,\ \overline{2} \vee 3,\ \overline{4}$	$\overline{1} \vee 2 \vee 4$	by ***T*-Conflict**
$1\ \overline{4} \bullet \overline{2}$	$1,\ \overline{2} \vee 3,\ \overline{4},\ \overline{1} \vee 2 \vee 4$	$\overline{1} \vee 2 \vee 4$	by **Learn**
$1\ \overline{4}$	$1,\ \overline{2} \vee 3,\ \overline{4},\ \overline{1} \vee 2 \vee 4$	no	by **Restart**
$1\ \overline{4}\ 2\ 3$	$1,\ \overline{2} \vee 3,\ \overline{4},\ \overline{1} \vee 2 \vee 4$	no	by **Propagate**$^{+}$
$1\ \overline{4}\ 2\ 3$	$1,\ \overline{2} \vee 3,\ \overline{4},\ \overline{1} \vee 2 \vee 4,\ \overline{1} \vee \overline{3} \vee 4$	$\overline{1} \vee \overline{3} \vee 4$	by ***T*-Conflict**, **Learn**
fail			by **Fail**

Modeling the Very Lazy Theory Approach

$$\underbrace{g(a)=c}_{1} \quad \land \quad \underbrace{f(g(a))\neq f(c)}_{\overline{2}} \lor \underbrace{g(a)=d}_{3} \quad \land \quad \underbrace{c\neq d}_{\overline{4}}$$

M	F	C	rule
	$1,\ \overline{2}\lor 3,\ \overline{4}$	no	
$1\ \overline{4}$	$1,\ \overline{2}\lor 3,\ \overline{4}$	no	by **Propagate**$^+$
$1\ \overline{4} \bullet \overline{2}$	$1,\ \overline{2}\lor 3,\ \overline{4}$	no	by **Decide**
$1\ \overline{4} \bullet \overline{2}$	$1,\ \overline{2}\lor 3,\ \overline{4}$	$\overline{1}\lor 2\lor 4$	by T-**Conflict**
$1\ \overline{4} \bullet \overline{2}$	$1,\ \overline{2}\lor 3,\ \overline{4},\ \overline{1}\lor 2\lor 4$	$\overline{1}\lor 2\lor 4$	by **Learn**
$1\ \overline{4}$	$1,\ \overline{2}\lor 3,\ \overline{4},\ \overline{1}\lor 2\lor 4$	no	by **Restart**
$1\ \overline{4}\ 2\ 3$	$1,\ \overline{2}\lor 3,\ \overline{4},\ \overline{1}\lor 2\lor 4$	no	by **Propagate**$^+$
$1\ \overline{4}\ 2\ 3$	$1,\ \overline{2}\lor 3,\ \overline{4},\ \overline{1}\lor 2\lor 4,\ \overline{1}\lor\overline{3}\lor 4$	$\overline{1}\lor\overline{3}\lor 4$	by T-**Conflict**, **Learn**
fail			by **Fail**

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$$\underbrace{g(a) = c}_{1} \quad \wedge \quad \underbrace{f(g(a)) \neq f(c)}_{\overline{2}} \vee \underbrace{g(a) = d}_{3} \quad \wedge \quad \underbrace{c \neq d}_{\overline{4}}$$

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	$1,\ \overline{2} \vee 3,\ \overline{4}$	no	
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$1\ \overline{4} \bullet \overline{2}$	$1,\ \overline{2} \vee 3,\ \overline{4}$	no	by **Decide**
$1\ \overline{4} \bullet \overline{2}$	$1,\ \overline{2} \vee 3,\ \overline{4}$	$\overline{1} \vee 2 \vee 4$	by T**-Conflict**
$1\ \overline{4} \bullet \overline{2}$	$1,\ \overline{2} \vee 3,\ \overline{4},\ \overline{1} \vee 2 \vee 4$	$\overline{1} \vee 2 \vee 4$	by **Learn**
$1\ \overline{4}$	$1,\ \overline{2} \vee 3,\ \overline{4},\ \overline{1} \vee 2 \vee 4$	no	by **Restart**
$1\ \overline{4}\ 2\ 3$	$1,\ \overline{2} \vee 3,\ \overline{4},\ \overline{1} \vee 2 \vee 4$	no	by **Propagate**$^+$
$1\ \overline{4}\ 2\ 3$	$1,\ \overline{2} \vee 3,\ \overline{4},\ \overline{1} \vee 2 \vee 4,\ \overline{1} \vee \overline{3} \vee 4$	$\overline{1} \vee \overline{3} \vee 4$	by T**-Conflict**, **Learn**
fail			by **Fail**

A Better Lazy Approach

The very lazy approach can be improved considerably with

- An *on-line* SAT engine,
  which can accept new input clauses on the fly
- an *incremental and explicating T*-solver,
  which can
  1. check the T-satisfiability of M as it is extended and
  2. identify a small T-unsatisfiable subset of M once M becomes T-unsatisfiable

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$$\underbrace{g(a) = c}_{1} \quad \wedge \quad \underbrace{f(g(a)) \neq f(c)}_{\overline{2}} \vee \underbrace{g(a) = d}_{3} \quad \wedge \quad \underbrace{c \neq d}_{\overline{4}}$$

M	F	C	rule
	$1,\ \overline{2} \vee 3,\ \overline{4}$	no	
$1\ \overline{4}$	$1,\ \overline{2} \vee 3,\ \overline{4}$	no	by **Propagate**$^+$
$1\ \overline{4} \bullet \overline{2}$	$1,\ \overline{2} \vee 3,\ \overline{4}$	no	by **Decide**
$1\ \overline{4} \bullet \overline{2}$	$1,\ \overline{2} \vee 3,\ \overline{4}$	$\overline{1} \vee 2$	by **T-Conflict**
$1\ \overline{4}\ 2$	$1,\ \overline{2} \vee 3,\ \overline{4}$	no	by **Backjump**
$1\ \overline{4}\ 2\ 3$	$1,\ \overline{2} \vee 3,\ \overline{4}$	no	by **Propagate**
$1\ \overline{4}\ 2\ 3$	$1,\ \overline{2} \vee 3,\ \overline{4}$	$\overline{1} \vee \overline{3} \vee 4$	by **T-Conflict**
fail			by **Fail**

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	$1,\ \overline{2} \vee 3,\ \overline{4}$	no	
$1\ \overline{4}$	$1,\ \overline{2} \vee 3,\ \overline{4}$	no	by **Propagate**$^{+}$
$1\ \overline{4} \bullet \overline{2}$	$1,\ \overline{2} \vee 3,\ \overline{4}$	no	by **Decide**
$1\ \overline{4} \bullet \overline{2}$	$1,\ \overline{2} \vee 3,\ \overline{4}$	$\overline{1} \vee 2$	by **T-Conflict**
$1\ \overline{4}\ 2$	$1,\ \overline{2} \vee 3,\ \overline{4}$	no	by **Backjump**
$1\ \overline{4}\ 2\ 3$	$1,\ \overline{2} \vee 3,\ \overline{4}$	no	by **Propagate**
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	$1,\ \overline{2} \vee 3,\ \overline{4}$	no	
$1\ \overline{4}$	$1,\ \overline{2} \vee 3,\ \overline{4}$	no	by **Propagate**$^{+}$
$1\ \overline{4} \bullet \overline{2}$	$1,\ \overline{2} \vee 3,\ \overline{4}$	no	by **Decide**
$1\ \overline{4} \bullet \overline{2}$	$1,\ \overline{2} \vee 3,\ \overline{4}$	$\overline{1} \vee 2$	by **T-Conflict**
$1\ \overline{4}\ 2$	$1,\ \overline{2} \vee 3,\ \overline{4}$	no	by **Backjump**
$1\ \overline{4}\ 2\ 3$	$1,\ \overline{2} \vee 3,\ \overline{4}$	no	by **Propagate**
$1\ \overline{4}\ 2\ 3$	$1,\ \overline{2} \vee 3,\ \overline{4}$	$\overline{1} \vee \overline{3} \vee 4$	by **T-Conflict**
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	$1,\ \overline{2}\vee 3,\ \overline{4}$	no	
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$1\ \overline{4}\bullet\overline{2}$	$1,\ \overline{2}\vee 3,\ \overline{4}$	$\overline{1}\vee 2$	by ***T*-Conflict**
$1\ \overline{4}\ 2$	$1,\ \overline{2}\vee 3,\ \overline{4}$	no	by **Backjump**
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	$1,\ \overline{2}\vee 3,\ \overline{4}$	no	
$1\ \overline{4}$	$1,\ \overline{2}\vee 3,\ \overline{4}$	no	by **Propagate**$^{+}$
$1\ \overline{4}\bullet\overline{2}$	$1,\ \overline{2}\vee 3,\ \overline{4}$	no	by **Decide**
$1\ \overline{4}\bullet\overline{2}$	$1,\ \overline{2}\vee 3,\ \overline{4}$	$\overline{1}\vee 2$	by **T-Conflict**
$1\ \overline{4}\ 2$	$1,\ \overline{2}\vee 3,\ \overline{4}$	no	by **Backjump**
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$1\ \overline{4}\ 2\ 3$	$1,\ \overline{2} \vee 3,\ \overline{4}$	$\overline{1} \vee \overline{3} \vee 4$	by **T-Conflict**
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	$1,\ \overline{2}\vee 3,\ \overline{4}$	no	
$1\ \overline{4}$	$1,\ \overline{2}\vee 3,\ \overline{4}$	no	by **Propagate**$^+$
$1\ \overline{4}\bullet\overline{2}$	$1,\ \overline{2}\vee 3,\ \overline{4}$	no	by **Decide**
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M	F	C	rule
	$1,\ \overline{2} \vee 3,\ \overline{4}$	no	
$1\ \overline{4}$	$1,\ \overline{2} \vee 3,\ \overline{4}$	no	by **Propagate**$^+$
$1\ \overline{4} \bullet \overline{2}$	$1,\ \overline{2} \vee 3,\ \overline{4}$	no	by **Decide**
$1\ \overline{4} \bullet \overline{2}$	$1,\ \overline{2} \vee 3,\ \overline{4}$	$\overline{1} \vee 2$	by **T-Conflict**
$1\ \overline{4}\ 2$	$1,\ \overline{2} \vee 3,\ \overline{4}$	no	by **Backjump**
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$1\ \overline{4}\ 2\ 3$	$1,\ \overline{2} \vee 3,\ \overline{4}$	$\overline{1} \vee \overline{3} \vee 4$	by **T-Conflict**
fail			by **Fail**

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	$1,\ \overline{2} \vee 3,\ \overline{4}$	no	
$1\ \overline{4}$	$1,\ \overline{2} \vee 3,\ \overline{4}$	no	by **Propagate**$^{+}$
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$1\ \overline{4}\ 2\ 3$	$1,\ \overline{2} \vee 3,\ \overline{4}$	$\overline{1} \vee \overline{3} \vee 4$	by ***T*-Conflict**
fail			by **Fail**

Lazy Approach – Strategies

Ignoring **Restart** (for simplicity), a common strategy is to apply the rules using the following priorities:

1. If a clause is falsified by the current assignment M, apply **Conflict**
2. If M is T-unsatisfiable, apply T**-Conflict**
3. Apply **Fail** or **Explain+Learn+Backjump** as appropriate
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Theory Propagation

With T**-Conflict** as the only theory rule, the theory solver is used just to validate the choices of the SAT engine

With T**-Propagate** and T**-Explain**, it can also be used to guide the engine's search [Tin02]

$$T\textbf{-Propagate}\quad \frac{l \in \mathrm{Lit}(\mathsf{F}) \quad \mathsf{M} \models_T l \quad l, \bar{l} \notin \mathsf{M}}{\mathsf{M} := \mathsf{M}\ l}$$

$$T\textbf{-Explain}\quad \frac{\mathsf{C} = l \vee D \quad \bar{l}_1, \ldots, \bar{l}_n \models_T \bar{l} \quad \bar{l}_1, \ldots, \bar{l}_n \prec_{\mathsf{M}} \bar{l}}{\mathsf{C} := l_1 \vee \cdots \vee l_n \vee D}$$

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$$\underbrace{g(a) = c}_{1} \quad \wedge \quad \underbrace{f(g(a)) \neq f(c)}_{\overline{2}} \vee \underbrace{g(a) = d}_{3} \quad \wedge \quad \underbrace{c \neq d}_{\overline{4}}$$

M	F	C	rule
	$1,\ \overline{2} \vee 3,\ \overline{4}$	no	
$1\ \overline{4}$	$1,\ \overline{2} \vee 3,\ \overline{4}$	no	by **Propagate**$^+$
$1\ \overline{4}\ \overline{2}$	$1,\ \overline{2} \vee 3,\ \overline{4}$	no	by T**-Propagate** $(1 \models_T \overline{2})$
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Modeling Modern Lazy SMT Solvers

At the core, current lazy SMT solvers are implementations of the transition system with rules

(1) **Propagate**, **Decide**, **Conflict**, **Explain**, **Backjump**, **Fail**

(2) T**-Conflict**, T**-Propagate**, T**-Explain**

(3) **Learn**, **Forget**, **Restart**

Basic DPLL Modulo Theories $\stackrel{\text{def}}{=}$ (1) + (2)

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Correctness

Updated terminology:

Irreducible state: state to which no Basic DPLL MT rules apply

Execution: sequence of transitions allowed by the rules and starting with $\mathsf{M} = \emptyset$ and $\mathsf{C} = \mathsf{no}$

Exhausted execution: execution ending in an irreducible state

Proposition (Soundness) For every exhausted execution starting with $\mathsf{F} = F_0$ and ending with fail, the clause set F_0 is T-unsatisfiable.

Proposition (Completeness) For every exhausted execution starting with $\mathsf{F} = F_0$ and ending with $\mathsf{C} = \mathsf{no}$, F_0 is T-satisfiable; specifically, M is T-satisfiable and $\mathsf{M} \models_{\mathrm{p}} F_0$.

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Proposition (Termination) Every execution in which

(a) **Learn/Forget** are applied only finitely many times and

(b) **Restart** is applied with increased periodicity

is finite.

Lemma Every exhausted execution ends with either $\mathsf{C} = \mathsf{no}$ or fail.

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DPLL(T) Architecture

The approach formalized so far can be implemented with a simple architecture named DPLL(T) [GHN+04, NOT06]

$$\text{DPLL}(T) = \text{DPLL}(X) \text{ engine} + T\text{-solver}$$

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$$\text{DPLL}(T) = \text{DPLL}(X) \text{ engine} + T\text{-solver}$$

DPLL(X):

- Very similar to a SAT solver, enumerates Boolean models
- Not allowed: pure literal, blocked literal detection, ...
- Required: incremental addition of clauses
- Desirable: partial model detection

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$$\text{DPLL}(T) = \text{DPLL}(X) \text{ engine} + T\text{-solver}$$

T-solver:

- Checks the T-satisfiability of conjunctions of literals
- Computes theory propagations
- Produces explanations of T-unsatisfiability/propagation
- Must be incremental and backtrackable

Reasoning by Cases in Theory Solvers

For certain theories, determining that a set M is T-unsatisfiable requires reasoning by cases.

Example: T = the theory of arrays.

$$M = \{\underbrace{r(w(a,i,x),j) \neq x}_{1},\ \underbrace{r(w(a,i,x),j) \neq r(a,j)}_{2}\}$$

$i = j$) Then, $r(w(a,i,x),j) = x$. Contradiction with 1.

$i \neq j$) Then, $r(w(a,i,x),j) = r(a,j)$. Contradiction with 2.

Conclusion: M is T-unsatisfiable

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Case Splitting

A *complete* T-solver reasons by cases via (internal) case splitting and backtracking mechanisms

An alternative is to lift case splitting and backtracking from the T-solver to the SAT engine

Basic idea: encode case splits as sets of clauses and send them as needed to the SAT engine for it to split on them [BNOT06]

Possible benefits:

- All case-splitting is coordinated by the SAT engine
- Only have to implement case-splitting infrastructure in one place
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Splitting on Demand

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Basic Scenario:

$$\mathsf{M} = \{\ldots,\ s = \underbrace{r(w(a,i,t),j)}_{s'},\ \ldots\}$$

- Main SMT module: "Is M $\mathcal{T}$-unsatisfiable?"
- $\mathcal{T}$-solver: "I do not know yet, but it will help me if you consider these *theory lemmas*:

$$s = s' \wedge i = j \rightarrow s = t, \quad s = s' \wedge i \neq j \rightarrow s = r(a,j)\text{"}$$

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Modeling Splitting on Demand

To model the generation of theory lemmas for case splits, add the rule

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$$\frac{\models_T \exists \mathbf{v}(l_1 \vee \cdots \vee l_n) \quad l_1, \ldots, l_n \in L_{\mathrm{S}} \quad \mathbf{v} \text{ vars not in } \mathsf{F}}{\mathsf{F} := \mathsf{F} \cup \{l_1 \vee \cdots \vee l_n\}}$$

where L_{S} is a finite set of literals dependent on the initial set of clauses (see [BNOT06] for a formal definition of L_{S})

Note: For many theories with a theory solver, there exists an appropriate finite L_{S} for every input F

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Now we can relax the requirement on the theory solver:

When $\mathsf{M} \models_{\mathrm{p}} \mathsf{F}$*, it must either*

- *determine whether* $\mathsf{M} \models_T \bot$ *or*
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The T-solver is required to determine whether $\mathsf{M} \models_T \bot$ only if all literals in L_{S} are defined in M

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Example — Theory of Finite Sets

$$F: \quad x = y \cup z \quad \wedge \quad y \neq \emptyset \vee x \neq z$$

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$x = y \cup z$	F	by **Propagate**$^+$
$x = y \cup z \; \bullet \; y = \emptyset$	F	by **Decide**
$x = y \cup z \; \bullet \; y = \emptyset \; x \neq z$	F	by **Propagate**
$x = y \cup z \; \bullet \; y = \emptyset \; x \neq z$	$F, (x = z \vee e \in x \vee e \in z),$ $(x = z \vee e \notin x \vee e \notin z)$	by $\mathcal{T}$**-Learn**
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$\mathcal{T}$-solver can make the following deductions at this point:

$$e \in x \quad \cdots \quad \Rightarrow \quad e \in y \cup z \quad \cdots \quad \Rightarrow \quad e \in y \quad \cdots \quad \Rightarrow \quad e \in \emptyset \quad \Rightarrow \quad \perp$$

This enables an application of $\mathcal{T}$**-Conflict** with clause

$$x \neq y \cup z \;\vee\; y \neq \emptyset \;\vee\; x = z \;\vee\; e \notin x \;\vee\; e \in z$$

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$$x \neq y \cup z \ \vee \ y \neq \emptyset \ \vee \ x = z \ \vee \ e \notin x \ \vee \ e \in z$$

Example — Theory of Finite Sets

$$F: \quad x = y \cup z \quad \land \quad y \neq \emptyset \lor x \neq z$$

M	F	rule
$x = y \cup z$	F	by **Propagate**$^+$
$x = y \cup z \;\bullet\; y = \emptyset$	F	by **Decide**
$x = y \cup z \;\bullet\; y = \emptyset\; x \neq z$	F	by **Propagate**
$x = y \cup z \;\bullet\; y = \emptyset\; x \neq z$	$F, (x = z \lor e \in x \lor e \in z),$ $(x = z \lor e \notin x \lor e \notin z)$	by **$\mathcal{T}$-Learn**
$x = y \cup z \;\bullet\; y = \emptyset\; x \neq z \;\bullet\; e \in x$	$F, (x = z \lor e \in x \lor e \in z),$ $(x = z \lor e \notin x \lor e \notin z)$	by **Decide**
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Correctness Results

Correctness results can be extended to the new rule.

Soundness: The new T**-Learn** rule maintains satisfiability of the clause set.

Completeness: As long as the theory solver can decide $\mathsf{M} \models_T \bot$ when all literals in L_{S} are determined, the system is still complete.

Termination: The system terminates under the same conditions as before. Roughly:

- Any lemma is (re)learned only finitely many times
- **Restart** is applied with increased periodicity

Combining Theories

Need for Combining Theories and Solvers

Recall: Many applications give rise to formulas like:

$$a \approx b + 2 \wedge A \approx \mathrm{write}(B, a+1, 4) \wedge$$
$$(\mathrm{read}(A, b+3) \approx 2 \vee f(a-1) \neq f(b+1))$$

Solving that formula requires reasoning over

- the theory of linear arithmetic (T_{LA})
- the theory of arrays (T_{A})
- the theory of uninterpreted functions (T_{UF})

Question: Given solvers for each theory, can we combine them modularly into one for $T_{\mathrm{LA}} \cup T_{\mathrm{A}} \cup T_{\mathrm{UF}}$?

Under certain conditions, we can do it with the Nelson-Oppen combination method [NO79, Opp80]

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Motivating Example (Convex Case)

Consider the following set of literals over $T_{\mathrm{LRA}} \cup T_{\mathrm{UF}}$ (T_{LRA}, linear real arithmetic):

$$\begin{aligned} f(f(x) - f(y)) &= a \\ f(0) &> a + 2 \\ x &= y \end{aligned}$$

First step: *purify* literals so that each belongs to a single theory

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$$f(f(x) - f(y)) = a \quad \Longrightarrow \quad \begin{array}{r} f(e_1) = a \\ e_1 = f(x) - f(y) \end{array} \quad \Longrightarrow \quad \begin{array}{r} f(e_1) = a \\ e_1 = e_2 - e_3 \\ e_2 = f(x) \\ e_3 = f(y) \end{array}$$

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$$f(0) > a + 2 \quad \Longrightarrow \quad \begin{array}{c} f(e_4) > a + 2 \\ e_4 = 0 \end{array} \quad \Longrightarrow \quad \begin{array}{c} f(e_4) = e_5 \\ e_4 = 0 \\ e_5 > a + 2 \end{array}$$

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Second step: exchange entailed *interface equalities*, equalities over shared constants $e_1, e_2, e_3, e_4, e_5, a$

L_1	L_2
$f(e_1) = a$	$e_2 - e_3 = e_1$
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Third step: check for satisfiability locally

Report unsatisfiable

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$f(x) = e_2$	$e_4 = 0$
$f(y) = e_3$	$e_5 > a + 2$
$f(e_4) = e_5$	$e_2 = e_3$
$x = y$	$a = e_5$
$e_1 = e_4$	

$L_1 \models_{\mathrm{UF}} e_2 = e_3$ $\quad$ $L_2 \models_{\mathrm{LRA}} e_1 = e_4$

$L_1 \models_{\mathrm{UF}} a = e_5$

Third step: check for satisfiability locally

$L_1 \not\models_{\mathrm{UF}} \perp$

$L_2 \models_{\mathrm{LRA}} \perp$

Report unsatisfiable

Motivating Example (Convex Case)

Second step: exchange entailed *interface equalities*, equalities over shared constants $e_1, e_2, e_3, e_4, e_5, a$

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$f(e_1) = a$	$e_2 - e_3 = e_1$
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Report unsatisfiable

Motivating Example (Non-convex Case)

Consider the following unsatisfiable set of literals over $T_{\text{LIA}} \cup T_{\text{UF}}$ (T_{LIA}, linear integer arithmetic):

$$
\begin{aligned}
1 \leq \quad x \quad & \leq 2 \\
f(1) \quad & = \quad a \\
f(2) \quad & = \quad f(1) + 3 \\
a \quad & = \quad b + 2
\end{aligned}
$$

First step: *purify* literals so that each belongs to a single theory

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$$f(1) = a \quad \Longrightarrow \quad \begin{aligned} f(e_1) &= a \\ e_1 &= 1 \end{aligned}$$

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$$f(2) = f(1) + 3 \quad \Longrightarrow \quad \begin{aligned} e_2 &= 2 \\ f(e_2) &= e_3 \\ f(e_1) &= e_4 \\ e_3 &= e_4 + 3 \end{aligned}$$

Motivating Example (Non-convex Case)

Second step: exchange entailed *interface equalities* over shared constants $x, e_1, a, b, e_2, e_3, e_4$

L_1	L_2
$1 \leq x$	$f(e_1) = a$
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No more entailed equalities, but $L_1 \models_{\text{LIA}} x = e_1 \lor x = e_2$

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Consider each case of $x = e_1 \vee x = e_2$ separately

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Case 1) $x = e_1$

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$L_2 \models_{\mathrm{UF}} a = b$, which entails $\perp$ when sent to L_1

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$e_2 = 2$	$x = e_2$
$e_3 = e_4 + 3$	
$a = e_4$	
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Case 2) $x = e_2$

Motivating Example (Non-convex Case)

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$L_2 \models_{\mathrm{UF}} e_3 = b$, which entails $\perp$ when sent to L_1

The Nelson-Oppen Method

- For $i = 1, 2$, let T_i be a first-order theory of *signature* Σ_i (set of function and predicate symbols in T_i other than $=$)
- Let $T = T_1 \cup T_2$
- Let $\mathcal{C}$ be a finite set of *free* constants (i.e., not in $\Sigma_1 \cup \Sigma_2$)

We consider only input problems of the form

$$L_1 \cup L_2$$

where each L_i is a finite set of *ground* (i.e., variable-free) $(\Sigma_i \cup \mathcal{C})$-literals

Note: Because of purification, there is no loss of generality in considering only ground $(\Sigma_i \cup \mathcal{C})$-literals

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The Nelson-Oppen Method

Bare-bones, non-deterministic, non-incremental version

[Opp80, Rin96, TH96]:

Input: $L_1 \cup L_2$ with L_i finite set of ground $(\Sigma_i \cup \mathcal{C})$-literals
Output: **sat** or **unsat**

1. Guess an *arrangement* A, i.e., a set of equalities and disequalities over $\mathcal{C}$ such that

$$c = d \in A \ \text{ or } \ c \neq d \in A \ \text{ for all } c, d \in \mathcal{C}$$

2. If $L_i \cup A$ is T_i-unsatisfiable for $i = 1$ or $i = 2$, return unsat

3. Otherwise, return **sat**

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Correctness of the NO Method

Proposition (Termination) The method is terminating.

(Trivially, because there is only a finite number of arrangements to guess)

Proposition (Soundness) If the method returns **unsat** for every arrangement, the input is $(T_1 \cup T_2)$-unsatisfiable.

(Because satisfiability in $(T_1 \cup T_2)$ is always preserved)

Proposition (Completeness) If $\Sigma_1 \cap \Sigma_2 = \emptyset$ and T_1 and T_2 are stably infinite, when the method returns **sat** for some arrangement, the input is $(T_1 \cup T_2)$-is satisfiable.

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Stably Infinite Theories

Def. A theory T is *stably infinite* iff every quantifier-free T-satisfiable formula is satisfiable in an infinite model of T

Many interesting theories are stably infinite:

- Theories of an infinite structure (e.g., integer arithmetic)
- Complete theories with an infinite model (e.g., theory of dense linear orders, theory of lists)
- Convex theories (e.g., EUF, linear real arithmetic)

Def. A theory T is *convex* iff, for any set L of literals
$L \models_T s_1 = t_1 \lor \cdots \lor s_n = t_n \implies L \models_T s_i = t_i$ for some i

Note: With convex theories, arrangements do not need to be guessed—they can be computed by (theory) propagation

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The Nelson-Oppen method has been extended to some classes of non-stably infinite theories [TZ05, RRZ05, JB10]

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SMT Solving with *Multiple* Theories

Let $T_1, \ldots, T_n$ be theories with respective solvers $S_1, \ldots, S_n$

How can we integrate all of them cooperatively into a single SMT solver for $T = T_1 \cup \cdots \cup T_n$?

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Quick Solution:

1. Combine $S_1, \ldots, S_n$ with Nelson-Oppen into a theory solver for T
2. Build a DPLL(T) solver as usual

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How can we integrate all of them cooperatively into a single SMT solver for $T = T_1 \cup \cdots \cup T_n$?

Better Solution [Bar02, BBC$^+$05b, BNOT06]:

1. Extend DPLL(T) to DPLL($T_1, \ldots, T_n$)
2. Lift Nelson-Oppen to the DPLL($X_1, \ldots, X_n$) level
3. Build a DPLL($T_1, \ldots, T_n$) solver

Modeling DPLL($T_1, \ldots, T_n$) Abstractly

- Let $n = 2$, for simplicity
- Let T_i be of signature Σ_i for $i = 1, 2$, with $\Sigma_1 \cap \Sigma_2 = \emptyset$
- Let $\mathcal{C}$ be a set of free constants
- Assume wlog that each input literal has signature $(\Sigma_1 \cup \mathcal{C})$ or $(\Sigma_2 \cup \mathcal{C})$ (no *mixed* literals)
- Let $\mathsf{M}|_i \stackrel{\text{def}}{=} \{(\Sigma_i \cup \mathcal{C})\text{-literals of } \mathsf{M} \text{ and their complement}\}$
- Let $\mathrm{I}(\mathsf{M}) \stackrel{\text{def}}{=} \{c = d \mid c, d \text{ occur in } \mathcal{C}, \mathsf{M}|_1 \text{ and } \mathsf{M}|_2\} \cup$
  $\{c \neq d \mid c, d \text{ occur in } \mathcal{C}, \mathsf{M}|_1 \text{ and } \mathsf{M}|_2\}$

  (*interface literals*)

Abstract DPLL Modulo Multiple Theories

Propagate, **Conflict**, **Explain**, **Backjump**, **Fail** (unchanged)

Decide $\dfrac{l \in \mathrm{Lit}(\mathsf{F}) \cup \mathrm{I}(\mathsf{M}) \quad l, \bar{l} \notin \mathsf{M}}{\mathsf{M} := \mathsf{M} \bullet l}$

Only change: decide on interface equalities as well

T**-Propagate** $\dfrac{l \in \mathrm{Lit}(\mathsf{F}) \cup \mathrm{I}(\mathsf{M}) \quad i \in \{1, 2\} \quad \mathsf{M} \models_{T_i} l \quad l, \bar{l} \notin \mathsf{M}}{\mathsf{M} := \mathsf{M}\ l}$

Only change: propagate interface equalities as well, but reason locally in each T_i

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Abstract DPLL Modulo Multiple Theories

T-Conflict

$$\frac{\mathsf{C} = \mathsf{no} \quad l_1, \ldots, l_n \in \mathsf{M} \quad l_1, \ldots, l_n \models_{T_i} \bot \quad i \in \{1, 2\}}{\mathsf{C} := \bar{l}_1 \vee \cdots \vee \bar{l}_n}$$

T-Explain

$$\frac{\mathsf{C} = l \vee D \quad \bar{l}_1, \ldots, \bar{l}_n \models_{T_i} \bar{l} \quad i \in \{1, 2\} \quad \bar{l}_1, \ldots, \bar{l}_n \prec_{\mathsf{M}} \bar{l}}{\mathsf{C} := l_1 \vee \cdots \vee l_n \vee D}$$

Only change: reason locally in each T_i

I-Learn

$$\frac{\models_{T_i} l_1 \vee \cdots \vee l_n \quad l_1, \ldots, l_n \in \mathsf{M}|_i \cup \mathrm{I}(\mathsf{M}) \quad i \in \{1, 2\}}{\mathsf{F} := \mathsf{F} \cup \{l_1 \vee \cdots \vee l_n\}}$$

New rule: for entailed disjunctions of interface literals

Abstract DPLL Modulo Multiple Theories

T-Conflict

$$\frac{\mathsf{C} = \mathsf{no} \quad l_1, \ldots, l_n \in \mathsf{M} \quad l_1, \ldots, l_n \models_{T_i} \bot \quad i \in \{1, 2\}}{\mathsf{C} := \bar{l}_1 \vee \cdots \vee \bar{l}_n}$$

T-Explain

$$\frac{\mathsf{C} = l \vee D \quad \bar{l}_1, \ldots, \bar{l}_n \models_{T_i} \bar{l} \quad i \in \{1, 2\} \quad \bar{l}_1, \ldots, \bar{l}_n \prec_{\mathsf{M}} \bar{l}}{\mathsf{C} := l_1 \vee \cdots \vee l_n \vee D}$$

Only change: reason locally in each T_i

I-**Learn**

$$\frac{\models_{T_i} l_1 \vee \cdots \vee l_n \quad l_1, \ldots, l_n \in \mathsf{M}|_i \cup \mathrm{I}(\mathsf{M}) \quad i \in \{1, 2\}}{\mathsf{F} := \mathsf{F} \cup \{l_1 \vee \cdots \vee l_n\}}$$

New rule: for entailed disjunctions of interface literals

$$F := \overbrace{f(e_1) = a}^{0} \wedge \overbrace{f(x) = e_2}^{1} \wedge \overbrace{f(y) = e_3}^{2} \wedge \overbrace{f(e_4) = e_5}^{3} \wedge \overbrace{x = y}^{4} \wedge$$
$$\underbrace{e_2 - e_3 = e_1}_{5} \wedge \underbrace{e_4 = 0}_{6} \wedge \underbrace{e_5 > a + 2}_{7}$$

$$\underbrace{e_2 = e_3}_{8} \quad \underbrace{e_1 = e_4}_{9} \quad \underbrace{a = e_5}_{10}$$

M	F	C	rule
	F	no	
0 1 2 3 4 5 6 7	F	no	by **Propagate**$^+$
0 1 2 3 4 5 6 7 8	F	no	by T-**Propagate** $(1, 2, 4 \models_{\mathrm{UF}} 8)$
0 1 2 3 4 5 6 7 8 9	F	no	by T-**Propagate** $(5, 6, 8 \models_{\mathrm{LRA}} 9)$
0 1 2 3 4 5 6 7 8 9 10	F	no	by T-**Propagate** $(0, 3, 9 \models_{\mathrm{UF}} 10)$
0 1 2 3 4 5 6 7 8 9 10	F	$\overline{7} \vee \overline{10}$	by T-**Conflict** $(7, 10 \models_{\mathrm{LRA}} \bot)$
fail			by **Fail**

Example — Convex Theories

$$F := \overbrace{f(e_1) = a}^{0} \wedge \overbrace{f(x) = e_2}^{1} \wedge \overbrace{f(y) = e_3}^{2} \wedge \overbrace{f(e_4) = e_5}^{3} \wedge \overbrace{x = y}^{4} \wedge$$
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0 1 2 3 4 5 6 7 8 9 10	F	$\overline{7} \vee \overline{10}$	by T-**Conflict** $(7, 10 \models_{\mathrm{LRA}} \perp)$
fail			by **Fail**

Example — Non-convex Theories

$$F := \begin{array}{l} \overbrace{f(e_1) = a}^{0} \ \wedge\ \overbrace{f(x) = b}^{1} \ \wedge\ \overbrace{f(e_2) = e_3}^{2} \ \wedge\ \overbrace{f(e_1) = e_4}^{3} \ \wedge \\ \underbrace{1 \leq x}_{4} \ \wedge\ \underbrace{x \leq 2}_{5} \ \wedge\ \underbrace{e_1 = 1}_{6} \ \wedge\ \underbrace{a = b + 2}_{7} \ \wedge\ \underbrace{e_2 = 2}_{8} \ \wedge\ \underbrace{e_3 = e_4 + 3}_{9} \end{array}$$

$$\underbrace{a = e_4}_{10} \quad \underbrace{x = e_1}_{11} \quad \underbrace{x = e_2}_{12} \quad \underbrace{a = b}_{13}$$

M	F	C	rule
	F	no	
$0 \cdots 9$	F	no	by **Propagate**$^{+}$
$0 \cdots 9\ 10$	F	no	by T**-Propagate** $(0,\ 3 \models_{\mathrm{UF}} 10)$
$0 \cdots 9\ 10$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by **l-Learn** $(\models_{\mathrm{LIA}} \overline{4} \vee \overline{5} \vee 11 \vee 12)$
$0 \cdots 9\ 10 \bullet 11$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by **Decide**
$0 \cdots 9\ 10 \bullet 11\ 13$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by T**-Propagate** $(0,\ 1,\ 11 \models_{\mathrm{UF}} 13)$
$0 \cdots 9\ 10 \bullet 11\ 13$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	$\overline{7} \vee \overline{13}$	by T**-Conflict** $(7,\ 13 \models_{\mathrm{UF}} \perp)$
$0 \cdots 9\ 10\ \overline{13}$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by **Backjump**
$0 \cdots 9\ 10\ \overline{13}\ \overline{11}$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by T**-Propagate** $(0,\ 1,\ \overline{13} \models_{\mathrm{UF}} \overline{11})$
$0 \cdots 9\ 10\ \overline{13}\ \overline{11}\ 12$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by **Propagate**
$\cdots$			(exercise)
fail	$\cdots$	$\cdots$	by **Fail**

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$0 \cdots 9\ 10$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by **I-Learn** $(\models_{\mathrm{LIA}} \overline{4} \vee \overline{5} \vee 11 \vee 12)$
$0 \cdots 9\ 10 \bullet 11$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by **Decide**
$0 \cdots 9\ 10 \bullet 11\ 13$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by T**-Propagate** $(0,\ 1,\ 11 \models_{\mathrm{UF}} 13)$
$0 \cdots 9\ 10 \bullet 11\ \underline{13}$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	$\overline{7} \vee \overline{13}$	by T**-Conflict** $(7,\ 13 \models_{\mathrm{UF}} \bot)$
$0 \cdots 9\ \underline{10}\ \underline{13}$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by **Backjump**
$0 \cdots 9\ \underline{10}\ \underline{13}\ 11$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by T**-Propagate** $(0,\ 1,\ \overline{13} \models_{\mathrm{UF}} \overline{11})$
$0 \cdots 9\ 10\ 13\ 11\ 12$	$F,\ 4 \vee 5 \vee 11 \vee 12$	no	by **Propagate**
$\cdots$			(exercise)
fail	$\cdots$	$\cdots$	by **Fail**

Example — Non-convex Theories

$$F := \begin{array}{l} \overbrace{f(e_1) = a}^{0} \;\wedge\; \overbrace{f(x) = b}^{1} \;\wedge\; \overbrace{f(e_2) = e_3}^{2} \;\wedge\; \overbrace{f(e_1) = e_4}^{3} \;\wedge \\ \underbrace{1 \leq x}_{4} \;\wedge\; \underbrace{x \leq 2}_{5} \;\wedge\; \underbrace{e_1 = 1}_{6} \;\wedge\; \underbrace{a = b + 2}_{7} \;\wedge\; \underbrace{e_2 = 2}_{8} \;\wedge\; \underbrace{e_3 = e_4 + 3}_{9} \end{array}$$

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$0 \cdots 9\ \underline{10}\ \underline{13}$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by **Backjump**
$0 \cdots 9\ \underline{10}\ \underline{13}\ 11$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by T**-Propagate** $(0,\ 1,\ \overline{13} \models_{\mathrm{UF}} \overline{11})$
$0 \cdots 9\ 10\ 13\ 11\ 12$	$F,\ 4 \vee 5 \vee 11 \vee 12$	no	by **Propagate**
$\ldots$			(exercise)
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$0 \cdots 9\ 10 \bullet 11$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by **Decide**
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$0 \cdots 9\ \underline{10}\ \underline{13}$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by **Backjump**
$0 \cdots 9\ \underline{10}\ \underline{13}\ 11$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by T**-Propagate** $(0,\ 1,\ \overline{13} \models_{\mathrm{UF}} \overline{11})$
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$0 \cdots 9\ 10\ \overline{13}\ \overline{11}$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by **T-Propagate** $(0,\ 1,\ \overline{13} \models_{UF} \overline{11})$
$0 \cdots 9\ 10\ \overline{13}\ \overline{11}\ 12$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by **Propagate**
$\cdots$			(exercise)
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Applications

Some Applications of SMT

Program Analysis and Verification

- Software Model Checking[2] (e.g. BLAST, SLAM)
- K-Induction-Based Model Checking[3] (e.g. Kind)
- Concolic or Directed Automated Random Testing[4] (e.g. CUTE, KLEE, PEX)
- Program Verifiers (e.g. VCC,[5] Why3[6])
- Translation Validation for Compilers (e.g. TVOC[7])

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[2] Jhala and Majumdar, **Software Model Checking**, ACM Computing Surveys 2009.

[3] Hagen and Tinelli, **Scaling Up the Formal Verification of Lustre Programs with SMT-Based Techniques**, FMCAD'08.

[4] Godefroid, Klarlund, and Sen, **DART: Directed Automated Random Testing**, PLDI '05

[5] Dahlweid, Moskal, Santen et al. **VCC: Contract-based modular verification of concurrent C**, ICSE '09.

[6] Bobot, Filliâtre, Marché, and Paskevich, **Why3: Shepherd Your Herd of Provers**, Boogie '11.

[7] Zuck, Pnueli, Goldberg, Barrett et al., **Translation and Run-Time Validation of Loop Transformations**, FMSD '05.

Some Applications of SMT

Non-verification Applications

- AI (e.g. Robot Task Planning[8])
- Biology (e.g. Analysis of Synthetic Biology Models[9])
- Databases (e.g. Checking Preservation of Database Integrity[10])
- Network Analysis (e.g. Checking Security of OpenFlow Rules[11])
- Scheduling (e.g. Rotating Workforce Scheduling[12])
- Security (e.g. Automatic Exploit Generation[13])
- Synthesis (e.g. Symbolic Term Exploration[14])

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[8] Witsch, Skubch, et al., **Using Incomplete Satisfiability Modulo Theories to Determine Robotic Tasks**, IROS '13.

[9] Yordanov and Wintersteiger, **SMT-based analysis of Biological Computation**, NFM '13.

[10] Baltopoulos, Borgström, and Gordon, **Maintaining Database Integrity with Refinement Types**, ECOOP '11.

[11] Son, Shin, Yegneswaran et al., **Model Checking Invariant Security Properties in OpenFlow**, ICC '13.

[12] Erkinger, **Rotating Workforce Scheduling as Satisfiability Modulo Theories**, Master's Thesis, TU Wien, 2013.

[13] Avgerinos, Cha, Rebert et al. **Automatic Exploit Generation**, CACM '14.

[14] Kneuss, Kuraj, Kuncak, and Suter, **Synthesis Modulo Recursive Functions**, OOPSLA '13.

New Theories

SMT users are clamouring for more capabilities

New theories in the pipeline

- Theory of *sets with cardinality*
- Theory of *floating-point numbers*
- Theory of *separation logic*

Going forward

- There is a huge opportunity to design and implement decision procedures for new *domain-specific theories*

Scalability

Plenty of room for performance improvements

- SMT innovations continue at both the system and algorithm level
- Example: With recent breakthroughs in *arithmetic* and *bit-vector* algorithms, CVC4 can solve many problems that were previously too hard (for any solver)
- *Parallel computing* still largely untapped

Google

- Ongoing collaboration with Google with ambitious goals for using symbolic execution on Google code
- Lots of interesting research questions about how to make use of Google's *massive resources* to apply symbolic execution and SMT solving on a *massive scale*

Tool integration using proofs

Skeptical proof assistants

- Tools like *Coq* and *Isabelle/HOL* are used extensively to verify systems and algorithms, despite their lack of automation
- SMT solvers cannot currently help because these tools do not trust external results

Idea: Produce independently checkable proofs

- We are instrumenting CVC4 to produce *proof certificates* in a formal proof framework called LFSC
- One goal: *replay proofs* in tools like Coq and Isabelle/HOL
- Collaboration with Andrew Appel at Princeton to support his *verified software toolchain*

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Verifying Neural Networks

Deep Learning can do Amazing Things

- Identify images
- Recognize speech
- Drive cars and airplanes

Safety-Critical applications: Need ways to analyze safety of neural network

- Problem: case anlaysis explosion (SMT and LP solvers blow up)
- Culprit: activation functions, e.g. Rectified Linear Unit (ReLU)
- Solution: Develop a *custom* theory solver for neural networks

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Verifying Neural Networks

Reluplex

- Simple SMT solver for neural networks [15]
- Extends simplex algorithm to reason about ReLU functions
- Result: Can prove properties of networks an order of magnitude larger than any previous method
- Dramatic reduction in case analysis required (from 2^{300} to 2^{30})

Case study: ACAS Xu

- Traffic collision avoidance algorithm for unmanned aircraft
- Neural network controller being considered by FAA
- Reluplex successfully used to prove safety properties for these networks

[15] Katz, Barrett, Dill, et al., **Reluplex: An Efficient SMT Solver for Verifying Deep Neural Networks**, CAV ’17.

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Agile Hardware Project

`http://aha.stanford.edu`

Collaboration with Prof. Hanrahan and Horowitz

- Build an open-source hardware flow
- Make it possible to do quick and incremental iterations of hardware designs

SMT solvers being used to

- automatically verify circuit transformations
- provide automatic and incremental place-and-route

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Next steps

- Bulid open-source SMT-based model-checking tools
- Develop or extend SMT theories for hardware verification
- Verify interoperation of hardware module interfaces

Summary

SMT solvers

- Provide *general-purpose* logical reasoning
- Can be customized for *domain-specific* reasoning
- *Enabler for formal methods*: automatic, expressive, scalable
- No shortage of *challenging research problems*
  - with immediate practical impact

More information

SMT resources

- SMT Survey Article: available at

  `http://theory.stanford.edu/~barrett/pubs/BKM14.pdf`
- SMT-LIB standards and library `http://smtlib.org`
- SMT Competition `http://smtcomp.org`
- SMT Workshop `http://smt-workshop.org`

CVC4

- Visit the CVC4 website: `http://cvc4.cs.nyu.edu`
- Contact a CVC4 team member
- We welcome questions, feedback, collaboration proposals

Suggested Readings

1. R. Nieuwenhuis, A. Oliveras, and C. Tinelli. **Solving SAT and SAT Modulo Theories: From an abstract Davis-Putnam-Logemann- Loveland procedure to DPLL(T)**. Journal of the ACM, 53(6):937-977, 2006.

2. R. Sebastiani. **Lazy Satisfiability Modulo Theories**. Journal on Satisfiability, Boolean Modeling and Computation 3:141-224, 2007.

3. S. Krstić and A. Goel. **Architecting Solvers for SAT Modulo Theories: Nelson-Oppen with DPLL**. In Proceeding of the Symposium on Frontiers of Combining Systems (FroCoS'07). Volume 4720 of LNCS. Springer, 2007.

4. C. Barrett, R. Sebastiani, S. Seshia, and C. Tinelli. **Satisfiability Modulo Theories**. In Handbook of Satisfiability. IOS Press, 2009.

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A SAT-based approach for solving formulas over boolean and linear mathematical propositions.
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[ACG00] Alessandro Armando, Claudio Castellini, and Enrico Giunchiglia. SAT-based procedures for temporal reasoning.
In S. Biundo and M. Fox, editors, *Proceedings of the 5th European Conference on Planning (Durham, UK)*, volume 1809 of *Lecture Notes in Computer Science*, pages 97–108. Springer, 2000

[AMP06] Alessandro Armando, Jacopo Mantovani, and Lorenzo Platania. Bounded model checking of software using SMT solvers instead of SAT solvers.
In *Proceedings of the 13th International SPIN Workshop on Model Checking of Software (SPIN'06)*, volume 3925 of *Lecture Notes in Computer Science*, pages 146–162. Springer, 2006

[Bar02] Clark W. Barrett. *Checking Validity of Quantifier-Free Formulas in Combinations of First-Order Theories*.
PhD dissertation, Department of Computer Science, Stanford University, Stanford, CA, Sep 2002

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